Multifractal spectra and precise rates of decay in homogeneous fragmentations.

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Abstract

We consider a mass-conservative fragmentation of the unit interval. Motivated by a result of Berestycki [3], the main purpose of this work is to specify the Hausdorff dimension of the set of locations having exactly an exponential decay. The study relies on an additive martingale which arises naturally in this setting, and a class of Lévy processes constrained to stay in a finite interval.

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1 Introduction.

Fragmentation appears in a wide range of phenomena in science and technology, such as degradation of polymers, colloids, droplets, rocks,... See the proceedings [12] for some applications in physics, for example [18] for computer science, [10] for mineral crushing, and works quoted in [3] for some further references. This work is a contribution to the study of

the rates of decay of fragments. More precisely, our aim is to investigate the set of locations which have an exact exponential decay (see (1) below for a precise definition).

Roughly a homogeneous fragmentation of intervals $F(t)$ can be seen as a family of nested open sets in $(0,1)$ such that each interval component is spill independently of the others, independently of the way that spill before, and with the same law as that of the initial fragmentation (up to spatial rescaling). We will suppose that no loss of mass occurs during the process.

Let $x \in (0,1)$ and $I_x(t)$ be the interval component of the fragmentation $F(t)$ which contains x, and $|I_x(t)|$ its length. Bertoin showed in [7] that if V is a uniform random variable on $[0,1]$ which is independent of the fragmentation, then $\xi(t) := -\log|I_V(t)|$ is a subordinator entirely determined by the fragmentation characteristics. By the SLLN for a subordinator, there exists v_{typ} such that $\frac{\xi(t)}{t} \to v_{typ}$ $a.s.$, which means that $|I_V(t)| \approx e^{-v_{typ}t}$. Berestycki [3] computed the Hausdorff dimension of the set

$$G_v := \left\{ x \in (0,1) \ : \ \lim_{t\to\infty} \frac{1}{t} \log|I_x(t)| = -v \right\}$$

for all $v > 0$. In this article we shall rather consider for some $0 < a < b$ the set

$$G_{(v,a,b)} := \left\{ x \in (0,1) \ : \ a \leq \liminf_{t\to\infty} e^{vt}|I_x(t)| \leq \limsup_{t\to\infty} e^{vt}|I_x(t)| \leq b \right\}. \tag{1}$$

Our goal is to compute the Hausdorff dimension of the set $G_{(v,a,b)}$. Our approach relies on some results on Lévy processes constrained to stay in a given interval.

Firstly we will recall background on fragmentations and Lévy processes. Secondly we will consider an additive martingale M which is naturally associated to the problem and obtain a criterion for uniform integrability. This is used in Section 4 to derive some limit theorems which may be of independent interest (see Engländer and Kyprianou [15] for a related approach in the setting of spatial branching processes). Finally we will compute the Hausdorff dimension of $G_{(v,a,b)}$ in Section 5.

2 Preliminaries.

2.1 Definition of fragmentation.

We will recall some facts about homogeneous interval fragmentations, which are mostly lifted from [3] , [7] and [8]. More precisely, we will consider fragmentations defined on the space $\mathcal{U}$ of open subsets of $(0,1)$. We shall use the fact that every element U of $\mathcal{U}$ has an interval decomposition, i.e. there exists a collection of disjoint open intervals $(J_i)_{i\in I}$, where the set of indices I can be finite or countable, such that $U = \cup_{i\in I} J_i$. Each interval component is viewed as a fragment.

A homogeneous interval fragmentation is a Markov process with values in the space $\mathcal{U}$ which enjoys two keys properties. First the branching property: different fragments

have independent evolutions. Second, the homogeneity property: up to an obvious spacial rescaling, the law of the fragment process does not depend on the initial length of the interval.

Specifically, if $\mathbb{P}$ stands for the law of the interval fragmentation F started from $F(0) = (0,1)$, then for $s,t \geq 0$ conditionally on the open set $F(t) = \underset{i\in I}{\cup} J_i(t)$, the interval fragmentation $F(t+s)$ has the same law as $F^1(s) \cup F^2(s) \cup ...$ where for each i, $F^i(s)$ is a subset of $J_i(t)$ and has the same distribution as the image of $F(s)$ by the homothetic map $(0,1) \to J_i(t)$.

2.2 Poissonian construction of the fragmentation.

Recall that $\mathcal{U}$ denotes the space of open subsets of $(0,1)$, and set $\mathbf{1} = (0,1)$. For $U \in \mathcal{U}$,

$$|U|^{\downarrow} := (u_1, u_2, ...)$$

will be the decreasing sequence of the interval component lengths of U. For $U = (a_1, b_1) \in \mathcal{U}$, we define the affine transformation $g_U : (0,1) \to U$ given by $g_U(x) = a_1 + x(b_1 - a_1)$.

In this article we will only consider proper fragmentations (which means that the Lebesgue measure of $F(t)$ is equal to 1). In this case, Basdevant [1] has shown that the law of the interval fragmentation F is completely characterized by the so-called dislocation measure ν (corresponding to the jump-component of the process) which is a measure on $\mathcal{U}$ which fulfills the conditions

$$\nu(\mathbf{1}) = 0,$$

$$\int_{\mathcal{U}} (1 - u_1)\nu(dU) < \infty, \tag{2}$$

and

$$\sum_{i=1}^{\infty} u_i = 1 \qquad for\ \nu - almost\ every\ U\ \in \mathcal{U}.$$

This last assumption is imposed by the hypothesis of length-conservation and means that when a sudden dislocation occurs, the total length of the intervals is unchanged. Specialists will notice that the erosion rates of the fragmentation c_r and c_l are here equal to 0 for the same reason.

We now recall the interpretation of sudden dislocations of the fragmentation process in terms of atoms of a Poisson point process (see [1], [2]). Let ν be a dislocation measure fulfilling the preceding conditions. Let $K = ((\Delta(t), k(t)), t \geq 0)$ be a Poisson point process with values in $\mathcal{U} \times \mathbb{N}$, and with intensity measure $\nu \otimes \sharp$, where $\sharp$ is the counting measure on $\mathbb{N}$. As in [2], we can construct a unique $\mathcal{U}$-valued process $F = (F(t), t \geq 0)$ started from $(0,1)$, with paths that jump only for times $t \geq 0$ at which a point $(\Delta(t), k(t))$ occurs, and then $F(t)$ is obtained by replacing the $k(t)$-interval $J_{k(t)}(t-)$ by $g_{J_{k(t)}(t-)}(\Delta(t))$. This point of view will be used in Section 3.

Some information about the dislocation measure ν and therefore about the distribution of the homogeneous fragmentation F is contained in the function:

$$\kappa(q) := \int_{\mathcal{U}} \left(1 - \sum_{j=1}^{\infty} u_j^{q+1}\right) \nu(dU) \qquad \forall q > \underline{p} \tag{3}$$

with $\underline{p}$ the smallest real number for which κ remains finite :

$$\underline{p} := \inf \left\{ p \in \mathbb{R} : \int_{\mathcal{U}} \sum_{j=2}^{\infty} u_j^{p+1} \nu(dU) < \infty \right\}.$$

We have that $-1 \leq \underline{p} \leq 0$ (because $\int_{\mathcal{U}} (1-u_1)\nu(dU) < \infty$ and $\sum_{i=1}^{\infty} u_i = 1$ for ν-almost every $U \in \mathcal{U}$).

This point of view is the same as in [3] and [7], which deal with ranked fragmentation instead of interval fragmentation. In the latter the space $\mathcal{U}$ is replaced by the space of mass partitions

$$\mathcal{S}^{\downarrow} := \left\{ x = (x_1, x_2, ...) \mid x_1 \geq x_2 \geq ... \geq 0 \; , \sum_{i=1}^{\infty} x_i \leq 1 \right\}.$$

For the precise link between these two fragmentations see [1].

2.3 An important subordinator.

Let $x \in (0,1)$ and $I_x(t)$ be the interval component of the random open set $F(t)$ which contains x, and $|I_x(t)|$ its length. Let V be a uniform random variable on $[0,1]$ which is independent of the fragmentation.

Bertoin showed in [7] that

$$\xi(t) := -\log |I_V(t)|, \quad t \geq 0, \tag{4}$$

is a subordinator, with Laplace exponent $\kappa(q)$ defined in (3) (i.e. $\mathbb{E}(e^{-\lambda \xi(t)}) = e^{-t\kappa(\lambda)}$ for all $\lambda > \underline{p}$). In order to interpret this as a Lévy-Khintchine formula, we introduce the measure

$$L(dx) := e^{-x} \sum_{j=1}^{\infty} \nu(-\log u_j \in dx), \qquad x \in (0, \infty).$$

It is easy to check that $\int \min(1,x) L(dx) < \infty$, thus L is the Lévy measure of a subordinator, and we can check that $\kappa(q) = \int_{(0,\infty)} (1 - e^{qx}) \, L(dx)$.

In this article we shall consider the Lévy process $Y_t = vt - \xi(t)$. In order to apply certain results to this process, we will need to assume that its one-dimensional distributions are absolutely continuous. Let L^{ac} be the absolutely continuous part of the measure L. Tucker has shown in [23] that

$$\int_{\mathbb{R}_+} \frac{1}{1+x^2} L^{ac}(dx) = \infty, \tag{5}$$

ensures the absolute continuity of one-dimensional distribution of the Lévy process evaluated at any $t > 0$. As $\int \min(1,x)L(dx) < \infty$, the condition (5) is equivalent to :

$$L^{ac}([0,\epsilon)) = \infty \quad \text{for any} \quad \epsilon > 0. \tag{6}$$

Let ν_1 be the image of the measure ν by the map $U \to u_1$ (recall that u_1 is the length of the longest interval component of the open set U) and ν_1^{ac} be the absolutely continuous part of the measure ν_1. Throughout this work we will make the following assumption, which is easily seen to imply (6) (in fact we can even show that the two are equivalent):

$$\nu_1^{ac}([0,\epsilon)) = \infty \quad \text{for any} \quad \epsilon > 0. \tag{7}$$

In the next subsection, we will give some results about Lévy processes that will be needed in the sequel, and apply for $Y_t = vt - \xi(t)$.

2.4 An estimate for completely asymmetric Lévy processes.

For the next sections, we will need some technical notions about completely asymmetric Lévy processes. Therefore we recall some facts mostly lifted from [4] and [6]. Let $Y = (Y_t)_{t\geq 0}$ be a Lévy process with no positive jumps and $(\mathcal{E}_t)_{t\geq 0}$ the natural filtration associated to $(Y_t)_{t\geq 0}$. The case where Y is the negative of a subordinator is degenerate for our purpose and therefore will be implicitly excluded in the rest of the article. The law of the Lévy process started at $x \in \mathbb{R}$ will be denoted by $\mathbf{P}_x$ (so bold symbols $\mathbf{P}$ and $\mathbf{E}$ refer to the Lévy process while $\mathbb{P}$ and $\mathbb{E}$ refer to the fragmentation), its Laplace transform is given by

$$\mathbf{E}_0(e^{\lambda Y_t}) = e^{t\psi(\lambda)}, \quad \lambda,\ t \geq 0,$$

where $\psi : \mathbb{R}_+ \to \mathbb{R}$ is called the Laplace exponent.

Let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be the right inverse of ψ (which exists because $\psi : \mathbb{R}_+ \to \mathbb{R}$ is convex with $\lim_{t\to\infty} \psi(\lambda) = \infty$), i.e. $\psi(\phi(\lambda)) = \lambda \quad \forall \lambda \geq 0$.

Let us recall some important features on the two-sided exit problem (which is completely solved in [6]). For $\beta > 0$ we denote the first exit time from $(0,\beta)$ by

$$T_\beta = \inf\{t : \ Y_t \notin (0,\beta)\}. \tag{8}$$

Let $W : \mathbb{R}_+ \to \mathbb{R}_+$ be the scale function, that is the unique continuous function with Laplace transform:

$$\int_0^\infty e^{-\lambda x} W(x)dx = \frac{1}{\psi(\lambda)} \quad , \ \lambda > \phi(0).$$

For $q \in \mathbb{R}$, let $W^{(q)} : \mathbb{R}_+ \to \mathbb{R}_+$ be the continuous function such that for every $x \in \mathbb{R}_+$

$$W^{(q)}(x) := \sum_{k=0}^{\infty} q^k W^{*k+1}(x),$$

where $W^{*n} = W * ... * W$ denotes the nth convolution power of the function W (for more details about this see [4] or [6]). So that

$$\int_0^\infty e^{-\lambda x} W^{(q)}(x) dx = \frac{1}{\psi(\lambda) - q} \quad , \ \lambda > \phi(q).$$

The next statement is about the asymptotic behavior of the Lévy process killed when it exits $(0, \beta)$ (point 1 and 2), which is taken from [6], and about the Lévy process conditioned to remain in $(0, \beta)$ (point 3, 4 and 5), which is taken from Theorem 3.1 (ii) and Proposition 5.1 (i) and (ii) in [20] :

Theorem 1 *Let us define the transition probabilities*

$$P_t(x, A) := \mathbf{P}_x(Y_t \in A, t < T_\beta) \text{ for } x \in (0, \beta) \text{ and } A \in \mathcal{B}((0, \beta)),$$

and the critical value

$$\rho_\beta := \inf\{q \geq 0 \ ; \ W^{(-q)}(\beta) = 0\}, \tag{9}$$

Suppose that the one-dimensional distributions of the Lévy process are absolutely continuous. Then the following holds:

1. $\rho_\beta \in (0, \infty)$ *and the function* $W^{(-\rho_\beta)}$ *is strictly positive on* $(0, \beta)$
2. *Let* $\Pi(dx) := W^{(-\rho_\beta)}(\beta - x)dx$*. For every* $x \in (0, \beta)$*:*

   $$\lim_{t \to \infty} e^{\rho_\beta t} P_t(x, .) = c W^{(-\rho_\beta)}(x) \Pi(.)$$

   in the sense of weak convergence, where

   $$c := \left(\int_0^\beta W^{(-\rho_\beta)}(y) W^{(-\rho_\beta)}(\beta - y) dy \right)^{-1}.$$

3. *The process*

   $$D_t := e^{\rho_\beta t} \ \mathbf{1}_{\{t < T_\beta\}} \ \frac{W^{(-\rho_\beta)}(Y_t)}{W^{(-\rho_\beta)}(x)} \tag{10}$$

   is a $(\mathbf{P}_x, (\mathcal{E}_t))$*-martingale.*

4. *The mapping* $(x, q) \mapsto W^{(q)}(x)$ *is of class* $\mathcal{C}^1$ *on* $(0, \infty) \times (-\infty, \infty)$*.*
5. *The mapping* $\beta \mapsto \rho_\beta = \inf\{q > 0 : W^{(-q)}(\beta) = 0\}$ *is strictly decreasing and of class* $\mathcal{C}^1$ *on* $(0, \infty)$*.*

Remark 1 *The definition of ρ_β is of course complicated, however in the simple case when Y is a standard Brownian motion, we have:*

$$\rho_\beta = \pi^2/\beta^2 \quad \textit{and} \quad W^{(-\rho_\beta)}(x) = \frac{\beta}{\pi}\sin\left(\frac{\pi}{\beta}x\right).$$

In the case where Y is a standard stable process, the mapping of $\beta \to \rho_\beta$ is depicted in [5]. We also point at the more explicit lower bound (see Lemma 5 in [6]):

$$\rho_a \geq 1/W(a),$$

Another lower bound will be given in Remark 4 below.

Remark 2 *The formula for the constant c in part 2. of Theorem 1 stems from the relation*

$$e^{\rho_\beta t}\frac{W^{(-\rho_\beta)}(y)}{W^{(-\rho_\beta)}(x)}P_t(x, dy) \underset{t\to\infty}{\sim} cW^{(-\rho_\beta)}(\beta - y)W^{(-\rho_\beta)}(y)dy.$$

Integrating over $(0, \beta)$ and using the fact that D_t is a martingale yields the given expression.

We also refer to the recent article of T. Chan and A. Kyprianou [13] for further properties of $W^{(-\rho_\beta)}$.

Now we have recalled the background that is needed to solve our problem.

3 An additive martingale.

Now we turn our attention to the main purpose of this article and consider a homogeneous interval fragmentation $(F(t), t \geq 0)$ and some real numbers $v > 0$ and $0 < a < b$. We are interested in the asymptotic set:

$$G_{(v,a,b)} = \left\{ x \in (0,1) \ : \ a \leq \liminf_{t\to\infty} e^{vt}|I_x(t)| \leq \limsup_{t\to\infty} e^{vt}|I_x(t)| \leq b \right\},$$

with $|I_x(t)|$ the length of the interval component of $F(t)$ which contains x.

In order to do that, we will have to consider first the non asymptotic set:

$$\Lambda_{(v,a,b)} = \left\{ x \in (0,1) \ : \ ae^{-vt} < |I_x(t)| < be^{-vt} \ \forall t \geq 0 \right\},$$

for $0 < a < 1 < b$.

In this section and in the next we will assume that $0 < a < 1 < b$.

We introduce some notation, that we will need in the rest of the article: define the set of the "good" intervals at time t as

$$G(t) := \{I_x(t) : \ x \in (0,1) \ \text{ and } \ ae^{-vs} < |I_x(s)| < be^{-vs} \ \ \forall \ s \leq t\}. \tag{11}$$

Let $(\mathcal{F}_t)_{t\geq 0}$ be the natural filtration of the interval fragmentation $(F(t), t \geq 0)$. Let $(\mathcal{G}_t)_{t\geq 0}$ be the enlarged filtration defined by $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(I_V(t))$ where V is a uniform variable independent of the fragmentation). We can remark that for all t we have $\mathcal{G}_t \subsetneq \mathcal{F}_t \vee \sigma\{V\}$, and $\mathcal{G}_\infty = \mathcal{F}_\infty \vee \sigma\{V\}$.

We recall that $\xi(t) = -\log |I_V(t)|$ is a subordinator. More precisely we are interested in the Lévy process with no positive jump $Y_t := vt - \xi(t) + \log(1/a)$, and use the results of preceding subsection for this Lévy process. We remark that its Laplace exponent $\psi(\lambda)$ is equal to $v\lambda - \kappa(\lambda)$, with κ defined in Subsection 2.3. Since we have supposed (7), the one-dimensional distributions of the Lévy process Y_t are absolutely continuous and we can apply Theorem 1.

For this Lévy process Y let

$$T := T_{\log(b/a)}$$

and

$$\rho := \rho_{\log(b/a)},$$

where T_β is defined in (8) and ρ_β is defined in (9). We stress that ρ depends on v, a, b and κ.

To simplify the notation, let also

$$h(t) := W^{(-\rho)}(t - \log a)\mathbf{1}_{\{t\in(\log a, \log b)\}}$$

for all $t \in \mathbb{R}$, and $h(-\infty) = 0$.

By rewriting (10) with the new notation we get a $(\mathcal{G}_t)$-martingale

$$D_t = e^{\rho t}\ \mathbf{1}_{\{t<T\}}\ \frac{h(vt + \log |I_V(t)|)}{h(0)}, \qquad t \geq 0.$$

If I is an interval component of $F(t)$, we define the "killed" interval $I^\dagger$ by $I^\dagger = I$ if I is good (i.e. $I \in G(t)$ with $G(t)$ defined in (11)), else by $I^\dagger = \emptyset$. Projecting the martingale D_t on the sub-filtration $(\mathcal{F}_t)_{t\geq 0}$, we obtain an additive martingale

$$M_t := \frac{e^{\rho t}}{h(0)} \int_0^1 h(vt + \log |I_x^\dagger(t)|)\ dx\ , \qquad t \geq 0.$$

We notice that if $y \in I_x(t)$, then $I_y(t) = I_x(t)$. Now we will consider the interval decomposition $(J_i(t), J_2(t), ...)$ of the open $F(t)$ (see subsection 2.1). We can rewrite M_t as:

$$M_t = \frac{e^{\rho t}}{h(0)} \sum_{i\in\mathbb{N}} h\left(vt + \log |J_i^\dagger(t)|\right) |J_i^\dagger(t)|. \tag{12}$$

We will use this expression in the rest of the article.

Finally, let the absorption time of M_t at 0 be

$$\begin{aligned} \zeta &:= \inf\{t : M_t = 0\} \\ &= \inf\{t : G(t) = \emptyset\}, \end{aligned}$$

with the convention $\inf\emptyset = \infty$.

Our first result is:

Theorem 2 *In the previous notation, with the assumptions (7) and if $v > \rho$ holds, then:*

1. *The martingale M_t is bounded in* $\mathrm{L}^2(\mathbb{P})$.

2. *Conditionally on $\zeta = \infty$, we have:* $\lim_{t\to\infty} M_t > 0$.

Remark 3 *We stress that as ρ depends on v, a, b and κ, the condition $v > \rho$ involves implicitly the parameters a and b. In particular it forces $b > 2a$, otherwise there would never be more than one "good" interval (as a fragment of size x will split into at least two different fragments and the smallest one will have a size at most equal to $x/2$), and as a consequence we would have $M_\infty = 0$ a.s., in contradiction with the uniform integrability of $M_.$.*

The proof of Theorem 2.1. is given in the appendix.

In order to prove Theorem 2.2 we will first introduce some notation, then prove two lemmas, and after we will conclude.

Let I be an interval of $(0,1)$. The law of the homogeneous interval fragmentation started at I will be denoted by $\mathbb{P}_I$. We remark that $\mathbb{P}_I(M_\infty = 0|\zeta = \infty)$ only depends on the length of I. Therefore we define

$$g(x) := \mathbb{P}_I(M_\infty = 0|\zeta = \infty),$$

where I is an interval such that $|I| = x$. Let N be the integer part of $(2b-a)/a$. As we assume $v > \rho$, we have necessarily $b > 2a$ (see Remark 3), thus $N \geq 2$. Let $\eta := (b-a)N^{-1}$. We remark that $\eta < a$ and $b - a = N\eta$. Denote the first time when there are at least two good intervals by

$$T^F := \inf\{t : \sharp G(t) \geq 2\},$$

with the convention $\inf\emptyset = \infty$. We notice that T^F is an $(\mathcal{F}_t)$ stopping time as $\sharp G(t)$ is $\mathcal{F}_t$-adapted.

Lemma 1 *In the previous notation, supposing that (7) and $v > \rho$ hold, we get: for every open interval I*

$$\mathbb{P}_I(T^F = \infty|\zeta = \infty) = 0.$$

Proof We notice that, as the martingale M_t is not identically 0 and is uniformly integrable, we have $\mathbb{P}_I(T^F = \infty|\zeta = \infty) < 1$ (because $M_\infty = 0$ when $T^F = \infty$).

Let I be an open interval such that $|I| \in (a,b)$, $t_0 := \log(2b/a)/v$ and $\epsilon := a^2/(2b^2)$. Thus

$$|I|(1-\epsilon) > a/2 \geq be^{-vt_0} \quad \text{and} \quad |I|\epsilon < b\epsilon \leq ae^{-vt_0}$$

therefore, if the dislocation of I produces at time t_0 an interval of length at least $|I|(1-\epsilon)$ then this interval is too large to be good and the remaining ones are too small to be good either. As a consequence we have

$$\mathbb{P}_I(M_{t_0}=0) \geq \mathbf{P}_{\log|I|}(e^{-\xi(t_0)} > e^{-\log|I|}(1-\epsilon)) = \mathbf{P}(\xi(t_0) < -\log(1-\epsilon)),$$

by the homogeneous property of the fragmentation. Moreover since $\xi(t)$ is a subordinator, we get $p := \mathbf{P}(\xi(t_0) < -\log(1-\epsilon)) > 0$, therefore

$$\mathbb{P}_I(M_{t_0}=0) \quad \geq \quad p > 0. \tag{13}$$

Additionally for every open interval I such that $|I| \in (a,b)$:

$$\mathbb{P}_I(\sharp G(t) = 1 \; \forall t \leq t_0) \leq 1 - \mathbb{P}_I(M_{t_0}=0) \leq 1-p.$$

Using the strong Markov property of the fragmentation and (13) we find by induction that for all $k \in \mathbb{N}$:

$$\mathbb{P}_I(\sharp G(t) = 1 \; \forall t \leq kt_0) \leq (1-p)^k.$$

Therefore

$$\lim_{t\to\infty} \mathbb{P}_I(\sharp G(s) = 1 \; \forall s \leq t) = 0$$

and as a consequence

$$\mathbb{P}_I(T^F = \infty | \zeta = \infty) = 0.$$

■

Lemma 2 *In the previous notation, supposing that (7) and $v > \rho$ hold, we get:*

$$\sup_{a<x<b} g(x) = \max_{1\leq k\leq N} g(a+k\eta),$$

where $N = \lfloor (2b-a)/a \rfloor$ and $\eta = (b-a)/N$.

Proof We will prove this lemma by induction.
The hypothesis of induction is for $n \leq N$:

$$(H)_n \quad : \qquad \sup_{x\in(a,a+n\eta)} g(x) = \max_{1\leq k\leq n} g(a+\eta k).$$

$*$ The case $n=1$: let I be an open interval such that $|I| \in (a, a+\eta)$. We work under $\mathbb{P}_I$ conditionally on "non-extinction" (which means conditionally on the event $\zeta = \infty$). Let

$$T^1 := \inf\{t \geq 0 | \; \exists J(t) \in G(t) \;\; : \;\; e^{vt}|J(t)| \notin (a, a+\eta)\},$$

with $G(t)$ defined in (11). The random time T^1 is an $(\mathcal{F}_t)$ stopping times. As the quantity $vt - (-\log |J(t)|)$ creeps upwards with probability equals to 1 and as $J(t) \in G(t)$ implies that $e^{vt}|J(t)| > a$, we get

$$T^1 = \inf\{t \geq 0 |\ \exists J(t) \in G(t)\ :\ e^{vt}|J(t)| = a + \eta\}.$$

Moreover by the choice of η we have $a+\eta < 2a$, which implies that there is at most one good interval whose length is always in $(a, a+\eta)$. Recall from Lemma 1 that $\mathbb{P}_I(T^F < \infty | \zeta = \infty) = 1$, thus

$$\mathbb{P}_I(T^1 < \infty | \zeta = \infty) = 1.$$

Using the strong Markov property at the stopping times T^1, we get

$$g(x) \leq g(a+\eta) \quad , \quad x \in (a, a+\eta),$$

thus $(H)_1$ holds.

$*$ The case $n+1$ (with $n+1 \leq N$): we suppose that the hypothesis of induction holds for all $k \leq n$.

Let I be an open interval such that $|I| \in (a+n\eta, a+(n+1)\eta)$. We work under $\mathbb{P}_I$ conditionally on "non-extinction". Let

$$T^n := \inf\{t \geq 0 |\ \exists J(t) \in G(t)\ :\ e^{vt}|J(t)| \notin (a+n\eta, a+(n+1)\eta)\},$$

with $G(t)$ defined in (11). The random time T^n is an $(\mathcal{F}_t)$ stopping times. As the quantity $e^{vt}|J(t)|$ grows only continuously, we get

$$T^n = \inf\{t \geq 0 |\ \exists J(t) \in G(t)\ :\ e^{vt}|J(t)| = a+(n+1)\eta \text{ or } e^{vt}|J(t)| \in (a, a+n\eta]\}.$$

Moreover by the choice of η we have $a+\eta < 2a$, which implies that there is at most one good interval which length is always in $(a+n\eta, a+(n+1)\eta)$. Additionally by Lemma 1, we get $\mathbb{P}_I(T^F < \infty | \zeta = \infty) = 1$, thus

$$\mathbb{P}_I(T^n < \infty | \zeta = \infty) = 1.$$

Using the strong Markov property at the stopping times T^n, we get

$$g(|I|) \quad \leq \quad \max\left(g(a+(n+1)\eta), \sup_{y \in (a, a+n\eta]} g(y)\right).$$

As this holds for every open interval I such that $|I| \in (a+n\eta, a+(n+1)\eta)$, by the hypothesis of induction, we have established $(H)_{n+1}$.

■

Proof[Proof of Theorem 2.2.] With Lemma 2, we get that there exists a integer k_0 in $[1, N]$ such that $g(a+\eta k_0) = \sup_{x \in (a,b)} g(x)$ (if two or more values of k, are possible, we choose the smallest one). Let x_0 be $a + \eta k_0$.

Additionally, with Lemma 1, we get $\mathbb{P}_{(0,x_0)}(T^F < \infty | \zeta = \infty) = 1$. Using the strong property of Markov for the stopping times T^F, and with $n \geq 2$ the random number of good intervals of the fragmentation at time T^F and with $\alpha_1, ..., \alpha_n$ the length of those intervals, we get:

$$g(x_0) \leq \mathbb{E}(g(\alpha_1)...g(\alpha_n)) \leq \mathbb{E}(g(x_0)^n) \leq g(x_0)^2.$$

As $g(x_0) < 1$ by the uniformly integrability of M_t, we get that $g(x_0) = 0$ and finally that $g \equiv 0$.

■

4 Limit theorems.

In this section, we establish two corollaries of Theorem 2, which will be useful in the sequel.

Bertoin and Rouault (Corollary 2 in [11]) proved that

$$\lim_{t\to\infty} \frac{1}{t} \log \sharp\{I_x(t) : \; ae^{-vt} < |I_x(t)| < be^{-tv}\} = C(v), \tag{14}$$

where $C(v) := (\Upsilon_v + 1)v - \kappa(\Upsilon_v)$ and Υ_v is the reciprocal of v by κ' i.e, $\kappa'(\Upsilon_v) = v$ for $v \in (v_{min}, v_{max})$. [1]

Here we deal with the more stringent requirement: $\forall s \leq t, \; |I_x(s)| \in (ae^{-sv}, be^{-sv})$, and the next proposition gives the rates that we find in that case.

Proposition 1 *In the notation of the previous sections, with the assumptions (7) and if $v > \rho$ we get that conditionally on $\zeta = \infty$ (i.e. M is not absorbed at 0, or in a equivalent way $\Lambda_{(v,a,b)} \neq \emptyset$):*

$$\lim_{t\to\infty} \frac{1}{t} \log \sharp G(t) = \; v - \rho \qquad a.s. \tag{15}$$

Before proving this corollary we make the following remark

Remark 4 *It is interesting to compare the estimate found by Bertoin and Rouault and the present one (of course we have not considered the same set, nevertheless the two estimates are related). For this we show that for all $v \in (v_{min}, v_{max})$ and a and b such that $\rho \geq v_{min}$ we have $C(v) \geq v - \rho$. In this direction we use results from [3] Section 1. Let $\Psi(p) := p\kappa'(p) - \kappa(p)$ for all $p > 0$ with κ' the derivative of κ (this function is well defined because of the definition of $\underline{p}$ in Section 2 and because $\underline{p} \leq 0$). For every $p > 0$, $\Psi'(p) = p\kappa''(p) \leq 0$ since κ is concave. As a consequence Ψ is decreasing. With the definition of Υ_v, we get that the function $v \in (v_{min}, v_{max}) \mapsto \Upsilon_v \in \mathbb{R}$ is decreasing, additionally $\Upsilon_{v_{min}} > 0$, therefore the function $v \in (v_{min}, v_{max}) \mapsto g(\Upsilon_v) \in \mathbb{R}$ is increasing. Moreover $\Psi(\Upsilon_v) = C(v) - v$, hence for all $v \in (v_{min}, v_{max})$:*

$$C(v) - v \geq C(v_{min}) - v_{min} = -v_{min}.$$

[1] Where v_{min} is the maximum of the function $p \mapsto \kappa(p-1)/p$ on $(\underline{p}+1, \infty)$ and $v_{max} := \kappa'(\underline{p}^+)$ (see [3]).

Additionally as $\rho \geq v_{min}$, we finally obtain:

$$\forall v \in (v_{min}, v_{max}) \qquad C(v) \geq v - \rho.$$

As a consequence, we have checked that the rate of growth of $\sharp G(t)$ (defined in (11)) is lower that of $\sharp\{I_x(t) : \ |I_x(t)| \in (ae^{-tv}, be^{-vt})\}$, which was of course expected.

Proof In this proof we work conditionally on $\zeta = \infty$ (i.e M is not absorbed at 0). Applying Theorem 2, we get $M_\infty > 0$. In order to show that (15) holds, we will first look at the lower bound of the inequality, and then at the upper bound.

• With the definition of M_t in (12), of $G(t)$ and of $J_i^\dagger(t)$ at the beginning of Section 3 and by the conditioning, there exists $t^{'} > 0$ such that for all $t \geq t^{'}$:

$$\frac{M_\infty}{2} \quad \leq \quad \frac{e^{\rho t}}{h(0)} \sum_{i \in \mathbb{N}} h(vt + \log(|J_i^\dagger(t)|)) \ |J_i^\dagger(t)| \leq \frac{e^{\rho t}}{h(0)} \sum_{i \in \mathbb{N}} C_4 \ be^{-vt} \ \mathbf{1}_{\{J_i(t) \in G(t)\}},$$

with C_4 as maximum of $h(.)$ on $[\log a, \log b]$. Hence for all $t \geq t^{'}$:

$$\sharp G(t) \geq \ e^{(v-\rho)t} \frac{h(0)}{2C_4 b} M_\infty,$$

and as a consequence, conditionally on $\zeta = \infty$,

$$\liminf_{t \to \infty} \frac{1}{t} \log \sharp G(t) \geq \ v - \rho. \tag{16}$$

• Secondly we will show the converse inequality.

Let $0 < a^{'} < a < 1 < b < b^{'}$, and $\rho^{'} := \rho_{\log(b^{'}/a^{'})}$. Denote the set of "good" intervals associated to $a^{'}$ and $b^{'}$ by:

$$G^{'}(t) := \{I_x(t) : \quad x \in (0,1) \ \text{ and } \ |I_x(s)| \ \in \ (a^{'} e^{-vs}, b^{'} e^{-vs}) \quad \forall \ s \leq t\}.$$

Let $M_t^{'}$ be the martingale defined at the beginning of Section 3 (and denoted there by M) associated to $a^{'}, b^{'}$ instead of a, b. Plainly, if M_t is not absorbed at 0, then a fortiori $M_t^{'}$ is not absorbed at 0 either. Additionally, since $\log(b^{'}/a^{'}) > \log(b/a)$, and $\rho_.$ is strictly decreasing (see Theorem 1.5), we get $v > \rho > \rho^{'}$ and we may apply Theorem 2 for $a^{'}, b^{'}$ instead of a, b. We get $\lim_{t \to \infty} M_t^{'} = M_\infty^{'} > 0$.

With the definition (12) of M_t and with an analogue of the function $h(t)$, namely $t \in \mathbb{R}$

$$\varphi(t) := W^{(-\rho^{'})}(t + \log(1/a^{'})) \mathbf{1}_{\{t \in (\log a^{'}, \log b^{'})\}},$$

we get:

$$M_\infty^{'} \quad = \quad \lim_{t \to \infty} \frac{e^{\rho^{'} t}}{\varphi(0)} \sum_{i \in \mathbb{N}} \varphi(vt + \log|J_i(t)|) |J_i(t)| \ \mathbf{1}_{\{J_i(t) \in G^{'}(t)\}}.$$

Therefore there exists $t^{'} > 0$ such that for every $t \geq t^{'}$

$$\begin{aligned} 2M_{\infty}^{'} &\geq \frac{e^{\rho^{'} t}}{\varphi(0)} \sum_{i\in\mathbb{N}} \varphi(vt + \log |J_i(t)|)|J_i(t)| \ \mathbf{1}_{\{J_i(t)\in G^{'}(t)\}} \\ &\geq \frac{e^{\rho^{'} t}}{\varphi(0)} \sum_{i\in\mathbb{N}} \varphi(vt + \log |J_i(t)|) \ a^{'} e^{-vt} \ \mathbf{1}_{\{J_i(t)\in G(t)\}}. \end{aligned}$$

Since $(ae^{-vt}, be^{-vt}) \subsetneq (a^{'}e^{-vt}, b^{'}e^{-vt})$, we get by Theorem 1.1, that for all $x \in [\log a, \log b]$: $\varphi(x) > 0$. Because $[\log a, \log b]$ is compact and $\varphi(.)$ is a continuous function,

$$\inf_{x\in[\log a, \log b]} \varphi(x) > 0.$$

Combining this with

$$C_5 := 2M_{\infty}^{'}\varphi(0)/\left(a^{'} \inf_{x\in[\log a,\log b]} \varphi(x)\right) < \infty,$$

we get for all $t \geq t^{'}$:

$$C_5 \geq e^{(\rho^{'}-v)t} \sum_{i\in\mathbb{N}} \mathbf{1}_{\{J_i(t)\in G(t)\}}$$

and thus

$$C_5 e^{(v-\rho^{'})t} \geq \sharp G(t).$$

Hence for all $a^{'}, b^{'}$ such that $0 < a^{'} < a < 1 < b < b^{'}$:

$$\limsup_{t\to\infty} \frac{1}{t} \log \sharp G(t) \leq v - \rho^{'}.$$

For $a^{'} \to a$ and $b^{'} \to b$ we get by the continuity of $\rho_{.}$:

$$\limsup_{t\to\infty} \frac{1}{t} \log \sharp G(t) \leq v - \rho.$$

■

Now we will give an other corollary, using the same method as that of Bertoin and Gnedin in [9]. We encode the configuration $J^{\dagger}(t) = \{|J_i^{\dagger}(t)|\}$ of the lengths of good intervals into the random measure

$$\sigma_t := \frac{e^{\rho t}}{h(0)} \sum_{i\in\mathbb{N}} h\left(vt + \log |J_i^{\dagger}(t)|\right) |J_i^{\dagger}(t)| \delta_{\log(1/a)+vt+\log |J_i^{\dagger}(t)|}$$

which has total mass M_t.

The associated mean measure σ_t^* is defined by the formula

$$\int_0^\infty f(x)\sigma_t^*(dx) = \mathbb{E}\left(\int_0^\infty f(x)\sigma_t(dx)\right)$$

which is required to hold for all compactly supported continuous functions f. Since M_t is a martingale, σ_t^* is a probability measure. More precisely the next proposition establishes the convergence of the mean measure σ_t^*, and then of σ_t itself.

Proposition 2 *In the notation of the previous sections, with the assumptions (7), and $v > \rho$ we get:*

1. *The measures σ_t^* converge weakly, as $t \to \infty$, to the probability measure*

$$\varrho(dy) := ch(y + \log a)h(\log(b) - y)dy$$

   where $c > 0$ is the constant that appears in Theorem 1.5.

2. *For any bounded continuous f*

$$L^2 - \lim_{t\to\infty} \int_0^\infty f(x)\sigma_t(dx) = M_\infty \int_0^\infty f(x)\varrho(dx). \tag{17}$$

Proof

1. Firstly we prove the convergence of the mean measures $\sigma_t^* \to \varrho$. Let f be a bounded continuous function. By definition we get:
   $\int_0^\infty f(y)\sigma_t^*(dy)$

$$\begin{aligned} &= \mathbb{E}\left(\int_0^1 f(\log(1/a) + vt + \log|I_x^\dagger(t)|)\frac{e^{\rho t}}{h(0)} h\left(vt + \log|I_x^\dagger(t)|\right) \mathbf{1}_{\{I_x^\dagger(t)\in G(t)\}} dx\right) \\ &= \mathbf{E}_{\log(1/a)}\left(f(Y_t)e^{\rho t}\frac{h(Y_t + \log a)}{h(0)}\mathbf{1}_{\{t<T\}}\right), \end{aligned}$$

   with the definition of Y_t. Thus by the definition of P_t in Theorem 1, we get

$$\int_0^\infty f(y)\sigma_t^*(dy) = \int_0^{\log(b/a)} f(y)\frac{h(y+\log a)}{h(0)}e^{\rho t}P_t(\log(1/a), dy).$$

   By Theorem 1.2, we get

$$\int_0^\infty f(y)\sigma_t^*(dy) \underset{t\to\infty}{\sim} c\int_0^{\log(b/a)} f(y)h(y+\log a)h(\log(b) - y)dy.$$

   Therefore the measure σ_t^* converge weakly to the probability measure ϱ.

2. Now we show that the scaled empirical measures induced by $J(t)$ converge in the L^2-sense to the random measure $M_\infty \varrho$.

   Let f_1 and f_2 be two continuous functions bounded from above by 1, and

$$
\begin{aligned}
S_t &= \sum_{i,j} f_1(\log(1/a) + vt + \log |J_i^\dagger(t)|) \frac{e^{\rho t}}{h(0)} h\left(vt + \log |J_i^\dagger(t)|\right) |J_i^\dagger(t)| \\
&\quad \times f_2(\log(1/a) + vt + \log |J_j^\dagger(t)|) \frac{e^{\rho t}}{h(0)} h\left(vt + \log |J_j^\dagger(t)|\right) |J_j^\dagger(t)|.
\end{aligned}
$$

   We need to show that

$$
\mathbb{E}\left(S_t\right) \to \left(\int_0^\infty f_1(x)\varrho(dx)\right) \left(\int_0^\infty f_2(x)\varrho(dx)\right) \mathbb{E}\left(M_\infty^2\right) \tag{18}
$$

   for f_1 and f_2 positive and bounded from above by 1. Indeed, suppose (18) is shown. Denote

$$
A_t = \sum_j f_1(\log(1/a) + vt + \log |J_i^\dagger(t)|) \frac{e^{\rho t}}{h(0)} h\left(vt + \log |J_j^\dagger(t)|\right) |J_j^\dagger(t)|.
$$

   Take $f_2 = 1$ to conclude from (18) that

$$
\lim_{t\to\infty} \mathbb{E}(A_t M_t) = \int_0^\infty f_1(x)\varrho(dx)\mathbb{E}\left(M_\infty^2\right).
$$

   Similarly, by setting $f_1 = f_2$ we get

$$
\lim_{t\to\infty} \mathbb{E}\left(A_t^2\right) = \left(\int_0^\infty f_1(x)\varrho(dx)\right)^2 \mathbb{E}\left(M_\infty^2\right).
$$

   Recalling that $\mathbb{E}(M_t^2) \to \mathbb{E}(M_\infty^2)$ and combining the above we get the desired

$$
\lim_{t\to\infty} \mathbb{E}\left[\left(A_t - M_t \int_0^\infty f_1(x)\varrho(dx)\right)^2\right] = 0.
$$

   To prove (18) let us replace t by $t+s$ and condition on $J^\dagger = (|J_i^\dagger(s)|)_{i\in\mathbb{N}}$. We have two cases: write $i \sim_s j$ for the case where at time $t+s$ two coexisting intervals $J_i^\dagger(t+s)$ and $J_j^\dagger(t+s)$ stem from the same interval at time s, and $i \nsim_s j$ for the case these intervals are not included into the same interval component at time s. Therefore, with the notation

$$
S_{t+s}^{(1)} := \mathbb{E}\left(\sum_{i\sim_s j} S_{t+s} \mid J^\dagger(s)\right) \quad \text{and} \quad S_{t+s}^{(2)} := \mathbb{E}\left(\sum_{i\nsim_s j} S_{t+s} \mid J^\dagger(s)\right)
$$

we get:

$$S^{(1)}_{t+s} + S^{(2)}_{t+s} = \mathbb{E}(S_{t+s} | J^{\dagger}(s)).$$

For the studies of $S^{(1)}_{t+s}$ we use the homogeneous property of the fragmentation and the notation $I_0 = (0, \log(b/a))$, and get

$$\begin{aligned}
&|S^{(1)}_{t+s}| \\
&\leq \sum_i |J^{\dagger}_i(s)|^2 e^{2\rho s} \mathbb{E}\left(\sum_j |J^{\dagger}_j(t)| e^{\rho t}\right)^2 \sup_{x\in I_0}\left(\frac{h(x+\log a)}{h(0)}\right)^2 \sup_{x\in I_0}|f_1(x)| \sup_{x\in I_0}|f_2(x)| \\
&\leq be^{(\rho-v)s} C_6,
\end{aligned}$$

with

$$C_6 := \sum_i |J^{\dagger}_i(s)| e^{\rho t} \mathbb{E}\left(\sum_j |J^{\dagger}_j(t)| e^{\rho t}\right)^2 \sup_{x\in I_0} h(x+\log(a))^2 \sup_{x\in I_0}|f_1(x)| \sup_{x\in I_0}|f_2(x)|/h(0)^2$$

which is finite because

$$\mathbb{E}\left(\sum_j |J^{\dagger}_j(t)| e^{\rho t}\right) = \mathbf{E}\left(\mathbf{1}_{\{t<T\}} e^{\rho t}\right) < \infty.$$

Thus $S^{(1)}_{t+s} \to 0$ as $s \to \infty$ uniformly in t.

Now we look at $S^{(2)}_{t+s}$. We introduce the notation $y_k = |J^{\dagger}_k(s)|$. Write $i \searrow k$ if the length $|J^{\dagger}_i(t+s)|$ stems from y_k. By independence, the intervals which are included in the interval with length y_k and those which are included in the interval with length y_l evolve independently, thus gathering the lengths $|J^{\dagger}_i(t+s)|$ by the ancestors at time s yields

$$S^{(2)}_{t+s} = \sum_{k\neq l}\left(\mathbb{E}\sum_{i\searrow k} ...\right)\left(\mathbb{E}\sum_{j\searrow l} ...\right).$$

On the other hand, by self-similarity and convergence of the mean measures

$$\begin{aligned}
&\mathbb{E}\left(\sum_{i\searrow k} e^{\rho s}\frac{h\left(vs+\log\left(y_k/a\right)\right)}{h(0)} y_k \ f_1(vt+\log(|J^{\dagger}_i(t)|)+vs+\log(y_k/a))\right. \\
&\left. e^{\rho t}\frac{h\left(vt+\log\left(|J^{\dagger}_i(t)|\right)+vs+\log(y_k/a)\right)}{h\left(vs+\log\left(y_k/a\right)\right)} |J^{\dagger}_i(t)| \ \middle| \ J^{\dagger}(s)\right) \\
&\underset{t\to\infty}{\to} \quad e^{\rho s}\frac{h\left(vs+\log\left(y_k/a\right)\right)}{h(0)} y_k \left(\int_0^{\infty} f_1(x)\varrho(dx)\right),
\end{aligned}$$

and

$$\mathbb{E}\left(\sum_{j\searrow l} e^{\rho s}\frac{h\left(vs+\log\left(y_l/a\right)\right)}{h(0)}y_l\ f_2(vt+\log(|J_j^\dagger(t)|)+vs+\log(y_l/a))\right.$$

$$\left. e^{\rho t}\frac{h\left(vt+\log\left(|J_j^\dagger(t)|\right)+vs+\log(y_l/a)\right)}{h\left(vs+\log\left(y_l/a\right)\right)}|J_j^\dagger(t)|\ \middle|\ J^\dagger(s)\right)$$

$$\underset{t\to\infty}{\to}\quad e^{\rho s}\frac{h\left(vs+\log\left(y_l/a\right)\right)}{h(0)}y_l\left(\int_0^\infty f_2(x)\varrho(dx)\right).$$

Therefore by dominated convergence

$$\mathbb{E}\left(S_{t+s}^{(2)}\right)\ \underset{s\to\infty}{\sim}\ \left(\int_0^\infty f_1(x)\varrho(dx)\right)\left(\int_0^\infty f_2(x)\varrho(dx)\right)\mathbb{E}\left(\sum_{k\neq l}\frac{e^{\rho s}}{h(0)}|J_k^\dagger(s)|\right.$$

$$\left. h\left(vs+\log(|J_k^\dagger(s)|/a)\right)\frac{e^{\rho s}}{h(0)}h\left(vs+\log(|J_l^\dagger(s)|/a)\right)|J_l^\dagger(s)|\right).$$

Moreover with $C_7:=b\sup_{x\in I_0}|h(x+\log a)|^2/h(0)^2$, we get

$$\mathbb{E}\left(\sum_k e^{2\rho s}\frac{h\left(vs+\log\left(|J_k^\dagger(s)|/a\right)\right)^2}{h(0)^2}|J_k^\dagger(s)|^2\right)\leq C_7\mathbb{E}\left(\sum_k e^{\rho s}|J_k^\dagger(s)|\right)e^{(\rho-v)s}$$

which goes to 0 when $s\to\infty$, as a consequence

$$\mathbb{E}\left(\sum_{k\neq l} e^{\rho s}\frac{h\left(vs+\log\left(|J_k^\dagger(s)|/a\right)\right)}{h(0)}|J_k^\dagger(s)|e^{\rho s}\frac{h\left(vs+\log\left(|J_l^\dagger(s)|/a\right)\right)}{h(0)}|J_l^\dagger(s)|\right)$$
$$\underset{s\to\infty}{\sim}\mathbb{E}\left(M_s^2\right).$$

∎

5 The Hausdorff dimension.

In this section we use the notation and definitions of the previous sections. We recall that $\rho=\rho_{\log(b/a)}$, where $\rho_.$ is define in (9). Let *dim* be the Hausdorff dimension. The aim of this section would be to proof the main theorem:

Theorem 3 ***: Multifractal spectrum.*** *Assume (7):*

- *if $\rho > v$ holds, then:*

$$G_{(v,a,b)} = \emptyset \quad a.s.$$

- *if $\rho < v$ holds, then:*

$$dim(G_{(v,a,b)}) = 1 - \rho/v \quad a.s. \tag{19}$$

Remark 5 *1. Berestycki in [3] has computed the Hausdorff dimension of the set*

$$G_v = \left\{ x \in (0,1) \mid \lim_{t\to\infty} \frac{1}{t} \log |I_x(t)| = -v \right\}.$$

He found that for $v \in (v_{min}, v_{max})$, $dim(G_v) = C(v)/v$ (with $C(v)$ defined at the beginning of section 4). In Remark 4 we have shown that for all $v \in (\max(v_{min}, \rho), v_{max})$ we have $C(v) \geq v - \rho$ and we can notice that the inequality is strict for $\rho > v_{min}$. As a consequence the set $G_{(v,a,b)}$ has a Hausdorff dimension smaller than that of G_v, and also smaller than that one could have infer from equality (14).

2. In the case $v > v_{typ}$, we have $Y_t/t \underset{t\to\infty}{\to} v - v_{typ} > 0 \quad a.s.$ and

$$\mathbf{P}_{\log(1/a)}(\inf\{t:\ Y_t \leq 0\} = \infty) > 0.$$

Thus $W^{(-q)}(\infty) = 0$ for all $q \geq 0$ and then $\lim_{\beta\to\infty} \rho_\beta = 0$. Moreover using the fact that, $\lim_{\beta\to 0} \rho_\beta = \infty$ and $\rho_.$ is decreasing, we get that for all $v > v_{typ}$, there exist a and b such that $\rho_{\log(b/a)} < v$ and thus the fact that the set of good intervals is not empty.

The proof of this theorem use the non-asymptotic set $\Lambda_{(v,a,b)}$. In particular the key of the proof is the next proposition:

Proposition 3 *Assume (7) and $0 < a < b < 1$:*

- *if $\rho > v$ holds, then:*

$$\Lambda_{(v,a,b)} = \emptyset \quad a.s.$$

- *if $\rho < v$ holds, then: $\mathbb{P}(\Lambda_{(v,a,b)} \neq \emptyset) > 0$, and conditionally on $\Lambda_{(v,a,b)} \neq \emptyset$,*

$$dim(\Lambda_{(v,a,b)}) = 1 - \rho/v. \tag{20}$$

Proof

1. Let $v > 0$ and a and b such that $v < \rho$. We define

$$N(t) := \sharp G(t),$$

with $G(t)$ defined in (11). We remark that

$$N(t) = \int_0^1 \frac{1}{|I_x(t)|} \mathbf{1}_{\{I_x(t)\in G(t)\}}(x)dx.$$

and in particular

$$\mathbb{E}(N(t)) \quad = \quad \mathbb{E}\left(\int_0^1 \frac{1}{|I_x(t)|}\mathbf{1}_{\{I_x(t)\in G(t)\}}(x)dx\right).$$

Additionally by (4), we get

$$\mathbb{E}(N(t)) \quad = \quad e^{vt}\ \mathbf{E}\left(e^{\xi(t)-vt}\ \mathbf{1}_{\{vs-\xi(s)-\log a\in(0,\log(b/a))\ \forall\ s\leq t\}}\right).$$

With the notation $Y_t = vt - \xi(t)$ and P_t defined in Theorem 1 we rewrite the previous equality as:

$$\begin{aligned}\mathbb{E}(N(t)) &= e^{vt}\ \mathbf{E}_{\log(1/a)}\left(e^{-Y_t-\log a}\ \mathbf{1}_{\{t<T\}}\right)\\ &= \frac{1}{a}e^{(v-\rho)t}\ \int_0^{\log(b/a)} e^{-y+\rho t}P_t(\log(1/a), dy).\end{aligned}$$

By Theorem 1.2 we get

$$\mathbb{E}(N(t)) \underset{t\to\infty}{\sim} \frac{1}{a}e^{(v-\rho)t}\ c\ h(0)\int_0^{\log(b/a)} e^{-y}\quad \Pi(dy),$$

with Π defined in Theorem 1.

Finally as the function $y \mapsto e^{-y}\ h(\log(b) - y)$ is continuous, the integral above is a finite constant. Thus if $\rho > v$ then $\lim_{t\to\infty}\mathbb{E}(N(t)) = 0$, from which one concludes that $\lim_{t\to\infty} N(t) = 0$, i.e. $\Lambda_{(v,a,b)} = \emptyset$ *a.s.*

2. Now we deal with the case where a and b are such that $v > \rho$. We work conditionally on $\Lambda_{(v,a,b)} \neq \emptyset$ (or, equivalently, on the event $\zeta = \infty$, which has a positive probability by Theorem 2).

   • Firstly, in order to prove the lower bound of the Hausdorff dimension of $\Lambda_{(v,a,b)}$, we will use the same method as Berestycki in [3] . We will divide this proof into three steps. Each step will begin with a star ($\star$). In the first step we will construct a subset $\underset{n\in\mathbb{N}}{\cap}\mathbb{G}_\delta(n)$ of $\Lambda_{(v,a,b)}$, which will be defined latter on (see (22)). In the second we shall obtain a lower bound of the Hausdorff dimension of this subset. In order to do that we will construct an increasing process indexed by $t \in (0, 1)$, which only increases on $\underset{n\in\mathbb{N}}{\cap}\mathbb{G}_\delta(n)$, and which is Hölder continuous. In the last step we will conclude.

   $\star$ As in [3] for $\delta > 0$ we define for all $n \in \mathbb{N}$, $H_\delta(n)$ as a multi-type branching process with each particle corresponding to a segment of $G(\delta n)$ and

$$G_\delta(n) := \underset{I\in H_\delta(n)}{\cup} I,$$

with $G(t)$ defined in (11) (i.e. $G_\delta(n) = G(\delta n)$).

We notice that the family $(G_\delta(n))_{n\in\mathbb{N}}$ is nested and that $\underset{n\in\mathbb{N}}{\cap} G_\delta(n) = \Lambda_{(v,a,b)}$.

Let $\epsilon > 0$, and fix $\epsilon^{'} > 0$ and $\eta > 0$ such that $\eta < \min(\epsilon, v-\rho)$. By Proposition 1, for this $\epsilon^{'} > 0$ and $\eta > 0$, we may find $t_0 > \max((1+|\log(1-\epsilon^{'})|)/(\epsilon-\eta), \log(2)/(v-\rho-\eta))$ such that for all $t > t_0$:

$$\mathbb{P}(|t^{-1}\log(\sharp G(t)) - (v-\rho)| > \eta | \zeta = \infty) < \epsilon^{'}.$$

For each $t > 0$, we consider a variable $\widetilde{\chi}(t)$ whose law is given by

$$\mathbb{P}(\widetilde{\chi}(t) = 0) = \epsilon^{''},$$

and

$$\mathbb{P}(\widetilde{\chi}(t) = \lfloor e^{[(v-\rho)-\eta]t} \rfloor) = 1 - \epsilon^{''},$$

where $\lfloor . \rfloor$ is the integer part and $\epsilon^{''} := \mathbb{P}(|t^{-1}\log(\sharp G(t)) - (v-\rho)| > \eta | \zeta = \infty) < \epsilon^{'}$. Moreover by using that for all $x \geq 2$: $\log(x) - 1 \leq \log(\lfloor x \rfloor)$, we notice that

$$|t^{-1}\log(\mathbb{E}(\widetilde{\chi}(t))) - (v-\rho)| \leq \eta + t^{-1}(|\log(1-\epsilon^{'})| + 1).$$

Plainly $\widetilde{\chi}(t)$ is stochastically dominated by $\sharp G(t)$. Exactly as in [3] we can construct a true Galton-Watson tree $\mathbb{H}$ by thinning H_δ where $\delta > t_0$. More precisely the offspring distribution of $\mathbb{H}$ is given by the law of $\widetilde{\chi}(\delta)$. Let $m := \mathbb{E}(\widetilde{\chi}(\delta))$ be the expectation of the number of children of a particle. Therefore, we get

(a)
$$|\delta^{-1}\log m - (v-\rho)| < \epsilon. \tag{21}$$

(b) The family
$$(\mathbb{G}(n) := \underset{I\in\mathbb{H}(n)}{\cup} I)_{n\in\mathbb{N}} \tag{22}$$
is nested. The $\mathbb{G}(n)$ is the union of the interval of the n generation of $\mathbb{H}$.

(c) $\underset{n\in\mathbb{N}}{\cap} \mathbb{G}(n) \subseteq \Lambda_{(v,a,b)}$.
This last point makes sense because we work conditionally on $\zeta = \infty$.

$\star$ We fix $\epsilon > 0$. We choose $\delta > t_0$ as shown above and consider the tree $\mathbb{H}$. We define $Z(n)$ as the number of nodes of $\mathbb{H}$ at height n. By the theory of Galton-Watson processes, as we are working conditionally on the event $\Lambda_{(v,a,b)} \neq \emptyset$, we have that almost surely

$$m^{-n}Z(n) \to \mathcal{W} > 0.$$

Let σ be a node of our tree (thus it is also a subinterval of $(0,1)$). Fix an interval $I \subset (0,1)$ and introduce

$$\begin{aligned} \mathbb{H}_I(n) &:= \{\sigma \in \mathbb{H}(n), \sigma \cap I \neq \emptyset\}, \\ Z_I(n) &:= \sharp\, \mathbb{H}_I(n). \end{aligned}$$

Define

$$x \to L_x := \lim_n m^{-n} Z_{(0,x)}(n), \quad x \in \ (0,1).$$

We will now state a lemma that we will use to conclude:

Lemma 3 *For each $\epsilon > 0$,*

(a) There exists a version $\widetilde{L}$ of $(L_x)_{x\in[0,1]}$ which is Hölder continuous of order α for any $\alpha < 1 - \rho/v - \epsilon$ for every $\epsilon > 0$.

(b) The process $\widetilde{L}$ only grows on the set $\underset{n\in\mathbb{N}}{\cap} \mathbb{G}(n)$.

Proof[Proof of Lemma 3.]

(a) Exactly as in [3] , we show the first point by verifying Kolmogorov's criterium (see [22] Theorem 2.1 p.26). Let $W(\sigma)$ be the "renormalized weight" of the tree rooted at σ, i.e.,

$$W(\sigma) := \lim_{n\to\infty} m^{-n}\ \sharp\{\sigma^{'} \in \mathbb{H}(|\sigma| + n), \sigma^{'} \subset \sigma\},$$

where $|\sigma|$ is the generation of σ.
By the definition of L we have for all $x > y \in (0,1)$:

$$|L_x - L_y| = \lim_{n\to\infty} m^{-n} Z_{(x,y)}(n), \quad x \in \ (0,1).$$

For any J open subinterval of $(0,1)$, let

$$\eta(J) := \sup\{n \in \mathbb{N} :\ e^{-v\delta n} \geq |J|\} = \lfloor -\log(|J|)/v\delta \rfloor.$$

For all x, y such that $x < y$ by the definition of $L_.$, we get:

$$\begin{aligned} &|L_x - L_y| \\ &= \lim_n m^{-\eta((x,y))} m^{-n+\eta((x,y))} \sum_{\sigma\in\mathbb{H}_{(x,y)}(\eta((x,y)))} \sharp\{\sigma^{'} \in \mathbb{H}(|\sigma| + n - \eta((x,y))), \sigma^{'} \subset \sigma\} \\ &\leq m^{-\eta((x,y))} \sum_{\sigma\in\mathbb{H}_{(x,y)}(\eta((x,y)))} W(\sigma), \end{aligned}$$

and by the definition of $\eta(.)$:

$$\begin{aligned}|L_x - L_y| &\leq e^{\log m(\frac{1}{v\delta}\log(y-x)+1)} \sum_{\sigma \in \mathbb{H}_{(x,y)}(\eta((x,y)))} W(\sigma) \\ &\leq m|x-y|^{1-\epsilon-\rho/v} \sum_{\sigma \in \mathbb{H}_{(x,y)}(\eta((x,y)))} W(\sigma),\end{aligned}$$

by using (21). Moreover by the definition of good intervals, we have that for each n the sizes of intervals in $\mathbb{H}(n)$ have a lower bound given by $ae^{-v\delta n}$, so $a|J|e^{-v\delta}$ is a lower bound for the sizes of the intervals of $\mathbb{H}(\eta(J))$, and thus $Z_J(\eta(J)) \leq e^{v\delta}/a$. Therefore for all $\gamma > 1$ and all $J \subset (0,1)$ we have:

$$\begin{aligned}\mathbb{E}\left((\sum_{\sigma \in \mathbb{H}_J(\eta(J))} W(\sigma))^\gamma \right) &\leq \mathbb{E}((W_1 + ... + W_{\lfloor e^{v\delta}/a \rfloor + 1})^\gamma) \\ &\leq \mathbb{E}((W_1 + ... + W_{\eta(J)+2})^\gamma) < \infty,\end{aligned}$$

where the W_i are i.i.d. with the same law as W. The finiteness comes from the existence of finite moments of all orders for W (see for example Theorem 3.4 p. 479 of Harris [17]).

(b) The second point is clear by the choice of L.

■

⋆ To prove that $dim\left(\underset{n\in\mathbb{N}}{\cap}\mathbb{G}(n)\right) \geq 1 - \rho/v - \epsilon$, it is enough to show that

$$\sum_i diam(U_i)^{1-\rho/v-\epsilon} > 0 \tag{23}$$

for any cover $\{U_i\}$ of $\underset{n\in\mathbb{N}}{\cap}\mathbb{G}(n)$, where $diam(U_i)$ is the diameter of U_i. Clearly, it is enough to assume that the $\{U_i\}$ are intervals, and by expanding them slightly and using the compactness of the closure of $\underset{n\in\mathbb{N}}{\cap}\mathbb{G}(n)$, we only need to check (23) if $\{U_i\}$ is a finite collection of open subintervals of $[0,1]$.

Let $\cup_{i=0}^N (l_i, r_i)$ be a cover of $\underset{n\in\mathbb{N}}{\cap}\mathbb{G}(n)$ (where the (l_i, r_i) are disjoints open intervals). Therefore

$$\sum_{i=1}^N |\tilde{L}_{r_i} - \tilde{L}_{l_i}| = \mathcal{W}.$$

Thus for all such covers with $\max_i(r_i - l_i)$ small enough

$$\mathcal{W} \leq k \sum_{i=0}^N (r_i - l_i)^{1-\rho/v-\epsilon}$$

and hence
$$dim(\Lambda_{(v,a,b)}) \geq dim(\underset{n\in\mathbb{N}}{\cap}\mathbb{G}(n)) \geq 1-\rho/v-\epsilon.$$
To get the lower bound of the Hausdorff dimension of $\Lambda_{(v,a,b)}$, we let ϵ tend to 0.

• Secondly, the upper bound for (20) is an easy corollary of the fact that the Hausdorff dimension is smaller than the box-counting dimension (see [16] p.36-43), using the cover $\underset{n\geq N}{\cup} \quad \underset{i\in\theta_{v,a,b}(n)}{\cup} J_i(n)$, with $\theta_{v,a,b}(t)=\{i\in\mathbb{N} \mid J_i(t)\in G(t)\}$ (with $G(t)$ defined in Section 3).

■

Then we have the next corollary, which deals with the general case for a and b:

Corollary 1 *For $t^{'}\geq 0$ set*
$$\Lambda_{(v,a,b)}(t^{'}) := \left\{x\in(0,1)\ :\ ae^{-vt}<|I_x(t)|<be^{-vt}\ \forall t\geq t^{'}\right\}.$$
Assume (7), $0<a<b$ and $\rho<v$, then
$$\mathbb{P}(\Lambda_{(v,a,b)}(t^{'})\neq\emptyset) \underset{t^{'}\to\infty}{\to} 1,$$
and
$$\mathbb{P}\left(dim(\Lambda_{(v,a,b)}(t^{'}))=1-\rho/v\ \middle|\ \Lambda_{(v,a,b)}(t^{'})\neq\emptyset\right)=1.$$

Proof

1. The first part of the proof is a consequence of the homogeneity of the fragmentation and of Proposition 3.

2. Fix $\rho^{'}>\rho$. As $\lim_{\beta\to 0}\rho_\beta=\infty$, and, by Theorem 1.5, the application $\beta\to\rho_\beta$ is continuous and strictly decreasing, therefore there exists $\beta_0\in(1,b/a)$ such that $\rho^{'}=\rho_{\log(\beta_0)}$. Let $\epsilon:=(\beta_0-1)/(1+\beta_0)$, $a^{'}:=1-\epsilon$, $b^{'}:=1+\epsilon$, $x_0:=(\beta_0+1)(a+b/\beta_0)/4$ (notice that $x_0\in(a,b)$) and
$$p_0 := \mathbb{P}(dim(\Lambda_{(v,a^{'},b^{'})})\geq 1-\rho_{\log(b^{'}/a^{'})}/v).$$
By Proposition 3, we get that $p_0>0$. We notice that by the choice of $a^{'}$ and of $b^{'}$, we have $\rho_{\log(b^{'}/a^{'})}=\rho_{\log(\beta_0)}=\rho^{'}$.
Let I be an interval of $(0,1)$. The law of the homogeneous interval fragmentation started at I will be denoted by $\mathbb{P}_I$. We remark that $\mathbb{P}_I(dim(\Lambda_{(v,a,b)})\geq 1-\rho^{'}/v)$ only depends on the length of I. Thus we define
$$g_{a,b}(x) := \mathbb{P}_I(dim(\Lambda_{(v,a,b)})\geq 1-\rho^{'}/v),$$

where I is an interval such that $|I| = x$.
Let $x \in (x_0 a^{'}, x_0 b^{'})$. We remark that by the choice of x_0 and as $1 < \beta_0 < b/a$ we have that $(x_0 a^{'}, x_0 b^{'}) \subset (a, b)$ and thus

$$g_{a,b}(x) \geq g_{x_0 a', x_0 b'}(x).$$

Moreover by the scaling property of the fragmentation we get that

$$g_{x_0 a', x_0 b'}(x) = \mathbb{P}(dim(\Lambda_{(v, a'/x, b'/x)}) \geq 1 - \rho_{\log((b'/x)/(a'/x))}/v) = p_0$$

Therefore

$$\inf_{x \in (x_0 a^{'}, x_0 b^{'})} g_{a,b}(x) \geq p_0. \tag{24}$$

Let

$$B(t) = \{i : \quad x_0 a^{'} < e^{vt}|J_i(t)| < x_0 b^{'}\} \ \ , \ n_t = \sharp B(t),$$

where $(J_1, J_2, ...)$ is the interval decomposition of $F(t)$.
Fix $t^{'} \geq 0$. By applying the Markov property at time $t^{'}$ we get that

$$\begin{aligned}
&\mathbb{P}(dim(\Lambda_{(v,a,b)}(t^{'})) < 1 - \rho^{'}/v)) \\
&\leq \mathbb{E}\left(\prod_{i \in B(t^{'})} \mathbb{P}_{J_i(t')}(dim(\Lambda_{(v, x_0 a', x_0 b')}) < 1 - \rho^{'}/v) \right) \\
&\leq \mathbb{E}((1 - p_0)^{n_{t'}}),
\end{aligned}$$

by using (24). Therefore as $p_0 > 0$, $n_{t'} \underset{t' \to \infty}{\longrightarrow} \infty$ (see (14)) and with the first part of the proof we can conclude.

■

Now we are able to proof our main result:

Proof[Proof of Theorem 3.]
Observe that for all $n \in \mathbb{N}$, we have

$$\Lambda_{(v,a,b)}(n) \subset G_{(v,a,b)} \subset \bigcap_{\epsilon > 0} \bigcup_{m \in \mathbb{N}} \Lambda_{(v, a-\epsilon, b+\epsilon)}(m). \tag{25}$$

We can notice that the second inclusion is actually an equality.

• First we consider the case where $\rho > v$. As the application $\beta \to \rho_\beta$ is continuous and strictly decreasing (see Theorem 1.5), there exists $\epsilon_0 > 0$ such that $v < \rho_{\log((b+\epsilon_0)/(a-\epsilon_0))} < \rho$. Moreover by (25)

$$G_{(v,a,b)} \subset \bigcup_{m \in \mathbb{N}} \Lambda_{(v, a-\epsilon_0, b+\epsilon_0)}(m),$$

therefore thanks to Proposition 3 and the homogeneous property of the fragmentation, we get the first part of the proof.

• Second we consider the case where $\rho < v$. Thanks the second inclusion and the corollary 1, we get that: for all $\epsilon \in (0, a)$,

$$dim(G_{(v,a,b)}) \leq dim(\underset{n\in\mathbb{N}}{\cup}\Lambda_{(v,a-\epsilon,b+\epsilon)}(n)) = \max_{n} dim(\Lambda_{(v,a-\epsilon,b+\epsilon)}(n)) \leq 1 - \rho_{\log(\frac{b+\epsilon}{a-\epsilon})}/v.$$

Then by the continuity of $\rho_{\cdot}$ (see Theorem 1.5), we get the uper bound of the Hausdorff dimension of $G_{(v,a,b)}$.

The lower bound of the Hausdorff dimension is a consequence of the first inclusion of (25), as $dim(\Lambda_{(v,a,b)}(n)) = 1 - \rho/v$ with a probability which goes to 1 when n goes to infinity.

■

6 Appendix

6.1 A partition fragmentation.

In this appendix we give a proof of Theorem 2.1. (Section 3).

For this, we use the method of Bertoin and Rouault in [11] for fragmentation, which goes back to Lyons and al. [21] for Galton-Watson processes, and tools taken from the article of Engländer, Harris and Kyprianou [14].

We first introduce the notations that we need and we define what a partition fragmentation Π is. Let $\mathcal{P}$ the space of partition of $\mathbb{N}$, and for every integer k, the block $\{1, ..., k\}$ is denoted by $[k]$. As in [11], we call discrete point measure on the space $\Omega := \mathbb{R}_+ \times \mathcal{P} \times \mathbb{N}$, any measure :

$$w = \sum_{(t,\pi,k)\in\mathcal{D}}^{\infty} \delta_{(t,\pi,k)},$$

where $\mathcal{D}$ is a subset of $\mathbb{R}_+ \times \mathcal{P} \times \mathbb{N}$ such that

$$\forall t^{'} \geq 0 \;\; \forall n \in \mathbb{N} \;\; \sharp \Big\{(t,\pi,k) \in \mathcal{D} \mid t \leq t^{'}, \pi_{|[n]} \neq ([n], \emptyset, \emptyset, ...), k \leq n\Big\} < \infty$$

and for all $t \in \mathbb{R}$

$$w(\{t\} \times \mathcal{P} \times \mathbb{N}) \in \{0, 1\}.$$

Starting from an arbitrary discrete point measure ω on $\mathbb{R}_+ \times \mathcal{P} \times \mathbb{N}$, we will construct a nested partition $\Pi = (\Pi(t), t \geq 0)$ (which means that for all $t \geq t^{'}$ $\Pi(t)$ is a finer partition of $\mathbb{N}$ than $\Pi(t^{'})$). We fix $n \in \mathbb{N}$, the assumption that the point measure ω is discrete enables us to construct a step path $(\Pi(t, n), t \geq 0)$ with values in the space of partitions of $[n]$, which only jumps at times t at which the fiber $\{t\} \times \mathcal{P} \times \mathbb{N}$ carries an atom of ω, say (t, π, k), such that $\pi_{|[n]} \neq ([n], \emptyset, \emptyset, ...)$ and $k \leq n$. In that case, $\Pi(t, n)$ is the partition obtained by replacing the $k-th$ block of $\Pi(t-, n)$, denoted $\Pi_k(t-, n)$, by the restriction $\pi_{|\Pi_k(t-,n)}$ of π to this block, and leaving the other blocks unchanged. Of course for all $t \geq 0$, $(\Pi(t, n), n \geq 0)$ is

compatible (i.e. for every n, $\Pi(n,t)$ is a partition of $[n]$ such that the restriction of $\Pi(n+1,t)$ to $[n]$ coincide with $\Pi(n,t)$), as a consequence, there exists a unique partition $\Pi(t)$, such that for all $n \geq 0$ we have $\Pi(t)_{|[n]} = \Pi(t,n)$. With the terminology of [7], it is shown in [11] that this process Π is a (partition valued) homogeneous fragmentation.

One says that a block $B \subset \mathbb{N}$ has an asymptotic frequency, if the limit

$$|B| := \lim_{n\to\infty} n^{-1} card(B \cap [n])$$

exists. When every block of some partition $\pi \in \mathcal{P}$ has an asymptotic frequency, we write $|\pi| = (|\pi_1|, ...)$ and then $|\pi|^{\downarrow} = (|\pi_1|^{\downarrow}, ...) \in \mathcal{S}^{\downarrow}$ for the decreasing rearrangement of the sequence $|\pi|$. In the case where some block of the partition π does not have an asymptotic frequency, we decide that $|\pi| = |\pi|^{\downarrow} = \partial$, where ∂ stands for some extra point added to $\mathcal{S}^{\downarrow}$. **We stress that the process of ranked asymptotic frequencies $|\Pi|^{\downarrow}$ is a ranked fragmentation.**

Moreover, let ν be the dislocation measure associated to this ranked fragmentation (see Subsection 2.2). According to Theorem 2 in [7], there exists a unique measure μ on $\mathcal{P}$ which is exchangeable (i.e. invariant by the action of finite permutations on $\mathcal{P}$), and such that ν is the image of μ by the map that associate the decreasing rearrangement $|\pi|^{\downarrow}$ of the sequence of the asymptotic frequencies of the blocks of π, to π. Thanks to exchangeability, we get that for all measurable function $f : [0,1] \to \mathbb{R}_+$ such that $f(0) = 0$.

$$\int_{\mathcal{P}} f(|\pi_1|)\mu(d\pi) = \int_{\mathcal{S}^*} \sum_{i=1}^{\infty} s_i f(s_i)\nu(ds).$$

We denote the sigma-field generated by the restriction to $[0,t] \times \mathcal{P} \times \mathbb{N}$ by $\mathcal{G}_0(t)$. So $(\mathcal{G}_0(t))_{t\geq 0}$ is a filtration, and the nested partitions $(\Pi(t), t \geq 0)$ are $(\mathcal{G}_0(t))_{t\geq 0}$-adapted. We define also the sigma-field $(\mathcal{F}_0(t))_{t\geq 0}$ generated by the decreasing rearrangement $|\Pi(r)|^{\downarrow}$ of the sequence of the asymptotic frequencies of the blocks of $\Pi(r)$ for $r \leq t$. Of course $(\mathcal{F}_0(t))_{t\geq 0}$ is a sub-filtration of $(\mathcal{G}_0(t))_{t\geq 0}$.

Let $\mathcal{G}_1(t)$ the sigma-field generated by the restriction of the discrete point measure w to the fiber $[0,t] \times \mathcal{P} \times \{1\}$. So $(\mathcal{G}_1(t), t \geq 0)$ is a sub-filtration of $(\mathcal{G}_0(t), t \geq 0)$, and the first block of Π is $(\mathcal{G}_1(t), t \geq 0)$-measurable. Let $\mathcal{D}_1 \subseteq \mathbb{R}_+$ be the random set of times $r \geq 0$ for which the discrete point measure has an atom on the fiber $\{r\} \times \mathcal{P} \times \{1\}$, and for every $r \in \mathcal{D}_1$, denote the second component of this atom by $\pi(r)$.

We define the probability measure $\mathbf{P}^{\updownarrow}$ as the h-transform of $\mathbf{P}$ based on the martingale D_t (defined in Theorem 1 (3)):

$$d\mathbf{P}^{\updownarrow}_x|_{\mathcal{E}_t} = D_t d\mathbf{P}_x|_{\mathcal{E}_t}. \tag{26}$$

To simplify the notation, as in the section 3 we define for all $t \in \mathbb{R}$ $h(t) = W^{(-\rho)}(t + \log(1/a))\mathbf{1}_{\{t\in(\log(a),\log(b))\}}$. This function is well defined thanks to Theorem 1.

Let $P_i(t)$ the block of $\Pi(t)$ which contains i at time t. Similarly as in Section 3, for a homogeneous fragmentation, we define the killed partition

$$\Pi^{\dagger}_j(t) = \Pi_j(t)\mathbf{1}_{\{\exists i\in\mathbb{N}^* |\ \Pi_j(t)=P_i(t);\ \forall s\leq t \quad |P_i(s)|\in(ae^{-vs},be^{-vs})\}}.$$

When we project the martingale D_t of (10) on the sub-filtration $(\mathcal{G}_0(t))_{t\geq 0}$, we obtain an additive martingale

$$\frac{e^{\rho t}}{h(0)} \sum_{i=1}^{\infty} h(vt + \log(|\Pi_i^{\dagger}(t)|)) \ |\Pi_i^{\dagger}(t)| \ .$$

As $|\Pi|$ is a ranked fragmentation with dislocation measure ν, this martingale is the same as this of Section 3. From now on, we denote this martingale by M_t too.

Observe that the projection (26) on the sub-filtration $\mathcal{G}_0(t)$ give the identity:

$$d\mathbb{P}_x^{\updownarrow}|_{\mathcal{G}_0(t)} \ = M_t d\mathbb{P}_x|_{\mathcal{G}_0(t)}.$$

Like in lemma 8 (ii) [11], with the probability measure $\mathbb{P}^{\updownarrow}$ we get:

Lemma 4 *Under $\mathbb{P}^{\updownarrow}$, the restriction of w to $\mathbb{R}_+ \times \mathcal{P} \times \{2, 3, ...\}$ has the same distribution as under $\mathbb{P}$ and is independent of the restriction to the fiber $\mathbb{R}_+ \times \mathcal{P} \times \{1\}$.*

It follows immediately from Theorem 1 that

Remark 6 *For $x \in [0, \log(b/a)]$, let $F_x(t) := \mathbf{E}_x \left(\ e^{\rho t}\mathbf{1}_{\{T>t\}\}}\right)$ for $t \in [0, \infty)$, then $F_x(t)$ converges when $t \to \infty$ to a finite limit, and $F_x(.) : [0, \infty) \to [0, \infty)$ is càdlàg. In particular we have*

$$\sup_{x\in[0,\log(b/a)]} \ \sup_{t\geq 0} |F_x(t)| < \infty.$$

Remark 7 *We have for all $t \geq 0$:*

$$|M_t - M_{t-}| \leq e^{(\rho - v)t} \frac{b^2}{ah(0)} \sup_{x\in[\log a,\log b]} h(x) \qquad a.s.$$

If $v > \rho$, there exists $0 < C^{'} < \infty$ such that

$$\sup_{t\geq 0} |M_t - M_{t-}| < C^{'} \quad a.s.$$

Let

$$c_t := \frac{e^{\rho t}}{h(0)} h\left(vt + \log(|\Pi_1^{\dagger}(t)|)\right) \ |\Pi_1^{\dagger}(t)|$$

and

$$d_t := \frac{e^{\rho t}}{h(0)} \sum_{i=2}^{\infty} h\left(vt + \log\left(|\Pi_i^{\dagger}(t)|\right)\right) \ |\Pi_i^{\dagger}(t)|.$$

Now we have the background that we need to study

$$M_t = c_t + d_t,$$

and we will show that M is bounded in $\mathrm{L}^2(\mathbb{P})$. In order to do that, as $\mathbb{E}(M_t^2) = \mathbb{E}^{\updownarrow}(M_t)$, it is enough to prove that

$$\lim_{t\to\infty} \mathbb{E}^{\updownarrow}(M_t) < \infty \ .$$

6.2 The proof of Theorem 2.1.

• First we show that $\lim_{t\to\infty}\mathbb{E}^{\updownarrow}(c_t)=0$.

With the subordinator $\xi(t):=-\log(|\Pi_1(t)|)$, whose Laplace exponent is κ (exactly the same as this defined in Subsection 2.3), with the Lévy Process $Y_t=vt-\xi(t)+\log(1/a)$, and $T:=T_{\log(b/a)}$ defined in (8) associated to this Lévy Process, under $\mathbf{P}^{\updownarrow}_{\log(1/a)}$ we get:

$$c_t \;=\; \frac{e^{(\rho-v)t}}{h(0)}W^{(-\rho)}(Y_t)\; e^{Y_t}\;\mathbf{1}_{\{t<T\}}.$$

As a consequence,

$$\mathbf{E}^{\updownarrow}_{\log(1/a)}\left(\frac{W^{(-\rho)}(Y_t)}{h(0)}\; e^{Y_t}\;\mathbf{1}_{\{t<T\}}\right)=\mathbf{E}_{\log(1/a)}\left(\frac{W^{(-\rho)}(Y_t)^2}{h(0)^2}\; e^{Y_t}\; e^{\rho t}\mathbf{1}_{\{t<T\}}\right)$$
$$\leq \sup_{x\in[\log a,\log b]}(h(x))^2\,\frac{b}{ah(0)^2}\;F_{\log(1/a)}(t)$$

which is bounded by a constant independent of t by Remark 6, and as $\rho<v$, we have $\lim_{t\to\infty}e^{(\rho-v)t}=0$. Therefore:

$$\lim_{t\to\infty}\mathbb{E}^{\updownarrow}(c_t)=0\;. \tag{27}$$

• Now we consider d_t. As shown in [11] with $B(r,j)=\{i\geq 2:\Pi_i(t)\subseteq\pi_j(r)\cap\Pi_1(r-)\}$, we get, for every $r\in[0,t]$ and $j\geq 2$, conditionally on $r\in\mathcal{D}_1$, $\Pi_1(r-)$ and $\pi_j(r)$, the partition $(\Pi_i(t):i\in B(r,j))$ can be written in the form $\tilde{\Pi}^{(j)}(t-r)_{|\pi_j(r)\cap\Pi_1(r-)}$. Here $(\tilde{\Pi}^{(j)})_{j\in\mathbb{N}}$ is a family of i.i.d. homogeneous fragmentations distributed as Π under $\mathbb{P}$ and independent of the sigma-field $\mathcal{G}_1(t)$. As a consequence:

$$\underset{i\geq 2}{\cup}\Pi_i(t)=\underset{j\geq 2}{\cup}\;\underset{r\in[0,t]\cap\mathcal{D}_1}{\cup}\;\tilde{\Pi}^{(j)}(t-r)_{|\pi_j(r)\cap\Pi_1(r-)}.$$

Moreover $|\pi_j(r)||\Pi_1(r-)|$ is $\mathcal{G}_1(t)$ measurable, and we have that for all $i\in\mathbb{N}$

$$|\tilde{\Pi}_i^{(j)}(t-r)_{|\pi_j(r)\cap\Pi_1(r-)}|=|\tilde{\Pi}_i^{(j)}(t-r)||\pi_j(r)||\Pi_1(r-)|$$

so that we get:

$$\begin{aligned}\mathbb{E}^{\updownarrow}(d_t|\mathcal{G}_1(t)) \;\leq\;& \frac{e^{\rho t}}{h(0)}C_8\sum_{r\in[0,t]\cap\mathcal{D}_1}\sum_{j=2}^{\infty}|\pi_j(r)||\Pi_1^{\dagger}(r-)|\mathbb{1}_{\{a\leq|\pi_j(r)||\Pi_1^{\dagger}(r-)|e^{vr}\leq b\}}\\ &\sum_{i=1}^{\infty}\mathbb{E}^{\updownarrow}\left(|\tilde{\Pi}_i^{(j)}(t-r)\mathbb{1}_{\{a\leq|\tilde{\Pi}_i^{(j)}(t'-r)|e^{v(t'-r)}|\pi_j(r)||\Pi_1^{\dagger}(r-)|e^{vr}\leq b\ \forall t'\in[r,t]\}}\;|\mathcal{G}_1(t)\right)\\ \leq\;& \frac{e^{\rho t}}{h(0)}C_8\sum_{r\in[0,t]\cap\mathcal{D}_1}\sum_{j=2}^{\infty}|\pi_j(r)||\Pi_1^{\dagger}(r-)|\mathbb{1}_{\{a\leq|\pi_j(r)||\Pi_1^{\dagger}(r-)|e^{vr}\leq b\}}\\ &\sum_{i=1}^{\infty}\mathbb{E}^{\updownarrow}\left(|\tilde{\Pi}_i^{(j)}(t-r)\mathbb{1}_{\{a/b\leq|\tilde{\Pi}_i^{(j)}(t'-r)|e^{v(t'-r)}\leq b/a\ \forall t'\in[r,t]\}}\;|\mathcal{G}_1(t)\right),\end{aligned}$$

with C_8 the maximum of $h(t)$ on the compact $[\log(a), \log(b)]$. As $\tilde{\Pi}$ is independent of the sigma-field $\mathcal{G}_1(t)$, $\tilde{\Pi}$ has the same distribution under $\mathbb{P}$ as under $\mathbb{P}^{\updownarrow}$. $\tilde{\Pi}^{(j)}$ is also distributed as Π under $\mathbb{P}$. Thus,

$$\begin{aligned}
&\sum_{i=1}^{\infty} \mathbb{E}^{\updownarrow}\left(|\tilde{\Pi}_i^{(j)}(t-r) \mathbb{1}_{\{a/b \le |\tilde{\Pi}_i^{(j)}(t'-r)| e^{v(t'-r)} \le b/a \ \forall t' \in [r,t]\}}| \,\Big|\, \mathcal{G}_1(t) \right) \\
&= \sum_{i=1}^{\infty} \mathbb{E}^{\updownarrow}\left(|\tilde{\Pi}_i^{(j)}(t-r)| \mathbb{1}_{\{a/b \le |\tilde{\Pi}_i^{(j)}(t'-r)| e^{v(t'-r)} \le b/a \ \forall t' \in [r,t]\}} \right).
\end{aligned}$$

Now we have by size-biased sampling:

$$\begin{aligned}
&\sum_{i=1}^{\infty} \mathbb{E}\left(e^{\rho(t-r)} |\Pi_i(t-r)| \mathbb{1}_{\{a/b \le |\Pi_i(t'-r)| e^{v(t'-r)} \le b/a \ \forall t' \in [r,t]\}} \right) \\
&= \mathbb{E}\left(\ e^{\rho(t-r)} \mathbf{1}_{\{t-r < \inf\{s:\ |\Pi_1(s)| \notin\ (\frac{a}{b} e^{-vs}, \frac{b}{a} e^{-vs})\}\}} \right) \\
&= \mathbf{E}_{\log(1/a)}\left(\ e^{\rho(t-r)} \mathbf{1}_{\{T_{2\log(b/a)} > t-r\}} \right),
\end{aligned}$$

as $\rho_.$ is decreasing $\rho_{2\log(b/a)} \le \rho$, thus

$$\begin{aligned}
&\sum_{i=1}^{\infty} e^{\rho(t-r)} \mathbb{E}^{\updownarrow}\left(|\tilde{\Pi}_i^{(j)}(t-r) \mathbb{1}_{\{a/b \le |\tilde{\Pi}_i(t'-r)| e^{v(t'-r)} \le b/a \ \forall t' \in [r,t]\}}| \,\Big|\, \mathcal{G}_1(t) \right) \\
&\le \mathbf{E}_{\log(1/a)}\left(\ e^{\rho_{2\log(b/a)}(t-r)} \mathbf{1}_{\{T_{2\log(b/a)} > t-r\}} \right).
\end{aligned}$$

Therefore with $F_x'(t) := \mathbf{E}_x\left(\ e^{\rho_{2\log(b/a)} t} \mathbf{1}_{\{T_{2\log(b/a)} > t\}} \right)$ and since $|\Pi_1^{\dagger}(r-)| = a e^{Y_{r-} - vr} \mathbb{1}_{\{r- < T\}}$ under $\mathbb{P}^{\updownarrow}_{\log(1/a)}$, we get:

$$\mathbb{E}^{\updownarrow}(d_t | \mathcal{G}_1(t)) \ \le \ a \frac{e^{(\rho - v)r} C_8}{h(0)} \sum_{r \in [0,t] \cap \mathcal{D}_1} \sum_{j=2}^{\infty} |\pi_j(r)| e^{Y_{r-}} \sup_{x \in [0, \log(b/a)]} \sup_{t \ge 0} |F_x'(t)|.$$

Moreover we have by definition $e^{Y_{r-}} \mathbb{1}_{\{r- < T\}} \le b/a$. We let

$$C_9 := \sup_{x \in [0, \log(b/a)]} \sup_{t \ge 0} |F_x'(t)| b C_8 / h(0)$$

according to Remark 6 we have $C_9 < \infty$. Thus

$$\mathbb{E}^{\updownarrow}(d_t | \mathcal{G}_1(t)) \ \le \ C_9 \sum_{r \in [0,t] \cap \mathcal{D}_1} \sum_{j=2}^{\infty} e^{(\rho - v)r} |\pi_j(r)|.$$

Under $\mathbb{P}$, the $\mathcal{G}_0(t-)$ predictable compensator of

$$A_t := \sum_{r \in [0,t] \cap \mathcal{D}_1} \sum_{j=2}^{\infty} e^{(\rho - v)r} |\pi_j(r)|$$

is

$$N_t := \int_0^t dr \int_{\mathcal{P}} \mu(ds) e^{(\rho - v)r} \sum_{j=2}^{\infty} |\pi_j|.$$

Additionally

$$\int_{\mathcal{P}} \mu(ds) \sum_{j=2}^{\infty} |\pi_j| = \int_{\mathcal{S}^*} \nu(ds) \sum_{i=1}^{\infty} s_i \left[(\sum_{j=1}^{\infty} s_j) - s_i \right].$$

As $\sum_{j=1}^{\infty} s_j = 1$ $\nu - a.s.$, we achieve:

$$\int_{\mathcal{P}} \mu(ds) \sum_{j=2}^{\infty} |\pi_j| \leq \int_{\mathcal{S}^*} \nu(ds) 2(1 - s_1),$$

which is finite by (2). Moreover as $\rho < v$, the term $e^{(\rho-v)r}$ is integrable on $[0, \infty)$, so that we have $\lim_{t\to\infty} N_t < \infty$.

As both $X_t := A_t - N_t$ and M_t are martingales, by Theorem 4.50 of [19], we get that

$$XM - [X, M] \quad \text{is a local martingale.}$$

A sequence $(\tau_n = (T(m, n))_{m\in\mathbb{N}})_{n\in\mathbb{N}}$ of adapted subdivisions is called a Riemann sequence if $\sup_{m\in\mathbb{N}}[T(m+1, n) \wedge t - T(m, n) \wedge t] \to 0$ for all $t \in \mathbb{R}_+$. By Theorem 4.47 of [19], for any Riemann sequence $\{\tau_n = (T(m, n))_{m\in\mathbb{N}}\}_{n\in\mathbb{N}}$ of adapted subdivisions, the processes $S_{\tau_n}(X, M)$ defined by

$$S_{\tau_n}(X, M)_t := \sum_{m\in\mathbb{N}} (X_{T(m+1,n)\wedge t} - X_{T(m,n)\wedge t})(M_{T(m+1,n)\wedge t} - M_{T(m,n)\wedge t})$$

converge to the process $[X, M]$, in measure, uniformly on every compact interval.

We will now bound $S_{\tau_n}(X, M)_t$ uniformly in t. As

$$S_{\tau_n}(X, M)_t \leq \sup_{l\in\mathbb{N}} |M_{T(l+1,n)\wedge t} - M_{T(l,n)\wedge t}| \sum_{m\in\mathbb{N}} |X_{T(m+1,n)\wedge t} - X_{T(m,n)\wedge t}| \tag{28}$$

we will first focus on $\sum_{m\in\mathbb{N}} |X_{T(m+1,n)\wedge t} - X_{T(m,n)\wedge t}|$:

$$\begin{aligned}
&\sum_{m\in\mathbb{N}} |X_{(m+1)/n\wedge t} - X_{T(m,n)\wedge t}| \\
&\leq \sum_{m\in\mathbb{N}} \left(\sum_{r\in[T(m,n)\wedge t, T(m+1,n)\wedge t]\cap\mathcal{D}_1} \sum_{j=2}^{\infty} e^{(\rho-v)r} |\pi_j(r)| + \int_{T(m,n)\wedge t}^{T(m+1,n)\wedge t} dr \int_{\mathcal{P}} \mu(ds) e^{(\rho-v)r} \sum_{j=2}^{\infty} |\pi_j| \right) \\
&\leq \sum_{r\in[0,t]\cap\mathcal{D}_1} \sum_{j=2}^{\infty} e^{(\rho-v)r} |\pi_j(r)| + \int_0^{\infty} dr \int_{\mathcal{P}} \mu(ds) e^{(\rho-v)r} \sum_{j=2}^{\infty} |\pi_j|.
\end{aligned}$$

Therefore by the previous study of A_t and N_t we get that there exist $C_{10} < \infty$ independent of t such that:

$$\lim_{n\to\infty} \mathbb{E}\left(\sum_{m\in\mathbb{N}} |X_{T(m+1,n)\wedge t} - X_{T(m,n)\wedge t}|\right) \leq C_{10} \quad \text{for all} \quad t.$$

Moreover

$$\lim_{n\to\infty} \sup_{l\in\mathbb{N}} |M_{T(l+1,n)\wedge t} - M_{T(l,n)\wedge t}| \leq \sup_{r\leq t} |M_r - M_{r-}|$$

is a.s. bounded by C' (see remark 7) independently of t. Consequently by (28)

$$\lim_{t\to\infty} \mathbb{E}([X,M]_t) < \infty.$$

Thus as $XM - [X,M]$ is a local martingale, we get that $\lim_{t\to\infty} \mathbb{E}^{\updownarrow}(d_t) < \infty$.

Finally according to (27), we get

$$\lim_{t\to\infty} \mathbb{E}^{\updownarrow}(M_t) = \lim_{t\to\infty} \mathbb{E}^{\updownarrow}(\mathbb{E}^{\updownarrow}(d_t + c_t|\mathcal{G}_1(t))) < \infty.$$

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